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PHASE SPACE ANALYSIS IN ANISOTROPIC OPTICAL SYSTEMS

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Abstract

From the minimal action principle follow the Hamilton equations of evolution for geometric optical rays in anisotropic media. As in classical mechanics of velocity-dependent potentials, the velocity and the canonical momentum are not parallel, but differ by an anisotropy vector potential, similar to that of linear electromagnetism. Descartes' well known diagram for refraction is generalized and a factorization theorem holds for interfaces between two anisotropic media.

1 Fermat's principle

Fermat's principle states that the light ray joining two points in an optical medium takes the path where it employs an extremal time [1]:

$$\delta \int_A^B dt = \delta \int_A^B ds\, n(\vec{q}(s), \dot{\vec{q}}(s)) = 0.$$

Here we denote by ds the *length* element along the ray $\vec{q}$, the ray direction by $\dot{\vec{q}} = \frac{d\vec{q}}{ds}$ and by n the *refractive index* of the medium. The refractive index characterizes the optical medium. Constant n indicates that the medium is *homogeneous* (invariant under translations) and *isotropic* (invariant under rotations). In *anisotropic media*, the refractive index depends also on the *direction* of the ray [2].

We use one of the Cartesian coordinates of $\mathcal{R}^3$ as the evolution parameter to describe the evolution of the ray $\vec{q} = \begin{pmatrix} \mathbf{q} \\ z \end{pmatrix}$. Defining $\mathbf{v} = \frac{d\mathbf{q}}{dz}$ with $ds = \frac{dz}{\sqrt{1-\dot{\mathbf{q}}^2}} = dz\,\sqrt{1+\mathbf{v}^2}$ we write Fermat's principle as [3]

$$\delta \int_{z_A}^{z_B} dz\, L(\mathbf{q}(z), z; \mathbf{v}(z)) = 0,$$

with the Lagrangian function $L(\mathbf{q}, z; \mathbf{v}) = \sqrt{1+\mathbf{v}^2}\, n(\mathbf{q}, z; \mathbf{v})$.

2 Evolution equations

The Euler-Lagrange equations that follow from the Fermat principle are [4]

$$\frac{d}{dz}\mathbf{p} = \frac{\partial L}{\partial \mathbf{q}},$$

where the canonical *momentum* is

$$\mathbf{p} = \frac{\partial L}{\partial \mathbf{v}} = \frac{n\mathbf{v}}{\sqrt{1+\mathbf{v}^2}} + \sqrt{1+\mathbf{v}^2}\,\frac{\partial n}{\partial \mathbf{v}} = n\dot{\mathbf{q}} + \mathbf{A}(\mathbf{q}, z, \dot{\mathbf{q}}),$$

and we define the *anisotropy vector*

$$\mathbf{A} = \sqrt{1+\mathbf{v}^2}\,\frac{\partial n}{\partial \mathbf{v}} = (1 - \dot{\mathbf{q}}\,\dot{\mathbf{q}}^T)\,\frac{\partial n(\mathbf{q}, z, \dot{\mathbf{q}})}{\partial \dot{\mathbf{q}}}.$$

We obtain the Hamilton evolution equations through the Legendre transformation

$$\frac{d\mathbf{q}}{dz} = \frac{\partial H}{\partial \mathbf{p}}, \qquad \frac{d\mathbf{p}}{dz} = -\frac{\partial H}{\partial \mathbf{q}},$$

with the Hamiltonian function

$$H(\mathbf{q}, z; \mathbf{p}) = \mathbf{p}\cdot\mathbf{v} - L(\mathbf{q}, z; \mathbf{v}) = -\sqrt{n^2 - |\mathbf{p}-\mathbf{A}|^2} + \frac{(\mathbf{p}-\mathbf{A})\cdot\mathbf{A}}{\sqrt{n^2 - |\mathbf{p}-\mathbf{A}|^2}}\,.$$

In anisotropic media, the three-vectors of ray direction $\dot{\vec{q}}$, momentum $\vec{p}$, and anisotropy $\vec{A}$, are thus characterized by:

$$\vec{p} = n\dot{\vec{q}} + \vec{A}(\vec{q}, \dot{\vec{q}}), \qquad p_z = -H, \qquad |\dot{\vec{q}}| = 1, \qquad |\vec{p} - \vec{A}| = n(\vec{q}, \dot{\vec{q}}),$$

i.e., we have the orthogonal decomposition of momentum $\vec{p}$ into ray direction $\dot{\vec{q}}$ and the anisotropy three-vector

$$\vec{A} = \nabla_{\dot{\vec{q}}}\, n\Big|_{|\dot{\vec{q}}|=1} = (1 - \dot{\vec{q}}\,\dot{\vec{q}}^T)\,\frac{\partial n}{\partial \dot{\vec{q}}}\bigg|_{|\dot{\vec{q}}|=1} = \hat{\mathbf{e}}_\theta\,\frac{\partial n}{\partial \theta} + \frac{\hat{\mathbf{e}}_\phi}{\sin\theta}\,\frac{\partial n}{\partial \phi}\,.$$

The anisotropy vector is orthogonal to the direction of ray propagation $\dot{\vec{q}}$.

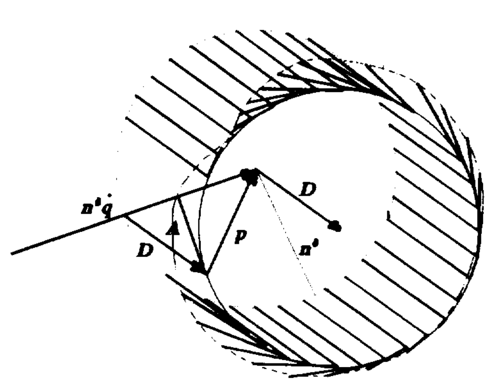
While $|\dot{\vec{q}}|$ sweeps over the ray direction sphere $\mathcal{S}_2$, the vector $\vec{p} - \vec{A}$ draws out a closed surface $n(\vec{q}, \dot{\vec{q}})$—the *ray surface* at the space point $\vec{q}$, and the three-vector $\vec{p}$ ranges correspondingly over another closed surface that we call the *Descartes ovoid* of the anisotropic medium at $\vec{q}$.

The Hamilton equations are thus written in manifestly euclidean-covariant form as

$$\frac{d\vec{q}}{dz} = \frac{\partial H}{\partial \vec{p}} = \frac{\vec{p} - \vec{A}}{p_z - A_z}, \qquad \frac{d\vec{p}}{dz} = -\frac{\partial H}{\partial \vec{q}} = \frac{n}{p_z - A_z}\,\frac{\partial n}{\partial \vec{q}}\,.$$

From the second equation it follows that $d\vec{p} \times \dfrac{\partial n}{\partial \vec{q}} = \vec{0}$. As in the isotropic case, we get the Ibn Sahl [5] Snell law of refraction between two anisotropic media [6].

FIGURE 1. Dipole medium: the momentum $\vec{p}$ of a ray is obtained by adding the direction vector $\dot{\vec{q}}$ times n^0 to the dipole vector of the medium $\vec{D}$. The anisotropy vector $\vec{A}$ ranges over a cardioid-type surface.



3 Dipole anisotropic media

Consider the refractive index with linear dependence on ray direction

$$\begin{aligned} n(\vec{q},\dot{\vec{q}}) &= n^0(\vec{q}) + D(\vec{q},\dot{\vec{q}}), \\ D(\vec{q},\dot{\vec{q}}) &= \sum_{j=x,y,z} D_j(\vec{q})\dot{q}_j = \vec{D}(\vec{q})^T\dot{\vec{q}}. \end{aligned}$$

We call n^0 the *monopole* part of the medium and $\vec{D}$ its *dipole* vector. The anisotropy vector is

$$\vec{A}^{(1)} = (1 - \dot{\vec{q}}\dot{\vec{q}}^T)\vec{D} = \vec{D} - \vec{D}^T(\dot{\vec{q}})\dot{\vec{q}} = (\dot{\vec{q}} \times \vec{D}) \times \dot{\vec{q}}.$$

This vector lies in the plane of $\dot{\vec{q}}$ and $\vec{D}$, and is orthogonal to the ray direction $\dot{\vec{q}}$. The relation between ray direction and optical momentum is $\vec{p} = n^0\dot{\vec{q}} + \vec{D}$. While $\dot{\vec{q}} \in \mathcal{S}_2$, the Descartes ovoid is a sphere of radius $n^0(\vec{q})$ and center at $\vec{D}$. (See Figure 1).

4 Quadrupole media

Consider now a refractive index with quadratic dependence on ray direction $\dot{\vec{q}}$

$$\begin{aligned} n(\vec{q},\dot{\vec{q}}) &= n^0(\vec{q}) + Q(\vec{q},\dot{\vec{q}}), \\ Q(\vec{q},\dot{\vec{q}}) &= \sum_{j,k=x,y,z} Q_{j,k}\,\dot{q}_j\dot{q}_k = \dot{\vec{q}}^T\hat{Q}\dot{\vec{q}}. \end{aligned}$$

We have a $\dot{\vec{q}}$-quadratic summand with coefficients $Q_{j,k}$ in a 3×3 *optical quadrupole matrix* $\hat{Q}$ that must be symmetric and traceless. It is common to restrict consideration to *principal axes*; in that frame of reference, $\hat{Q} = \mathrm{diag}\,(Q_x, Q_y, Q_z)$ and $Q_x + Q_y + Q_z = 0$. Then, the anisotropy and momentum vectors are

$$\vec{A}^{(2)} = 2(1 - \dot{\vec{q}}\dot{\vec{q}}^T)\hat{Q}\dot{\vec{q}} = 2[\hat{Q}\dot{\vec{q}} - Q(\vec{q},\dot{\vec{q}})\,\dot{\vec{q}}],$$

$$\vec{p} = [n(\vec{q},\dot{\vec{q}}) + 2(1 - \dot{\vec{q}}\dot{\vec{q}}^T)\hat{Q}]\dot{\vec{q}} = (n^0 + 2\hat{Q} - \dot{\vec{q}}^T\hat{Q}\dot{\vec{q}})\dot{\vec{q}}.$$

In two-dimensional optics, 2×2 symmetric traceless matrices have two independent coefficients that describe the ellipticity and orientation of the figure. When $\dot{\vec{q}}$ ranges over the sphere of directions $\mathcal{S}_2$, $\vec{p}$ will range over the Descartes ovoid of the quadrupole medium. (See Figure 2).

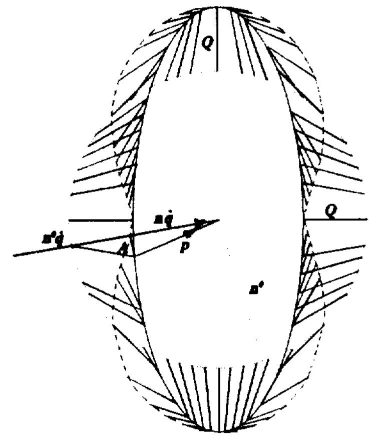


FIGURE 2. $n^0\dot{\vec{q}}$ ranges over the sphere (circle in the two dimensions of the figure) of radius n^0, $n\dot{\vec{q}}$ over the peanut-shaped surface, and the momentum $\vec{p}$ draws a Descartes oval. The thin lines joining points on the circle and on the oval relate the direction of the ray with the corresponding direction of the momentum vector.

5 Free propagation in homogeneous uniaxial media

For the uniaxial quadrupole media we can write the refractive index as

$$n(\dot{\vec{q}}) = n^0 + (\dot{x}, \dot{y}, \dot{z}) \begin{pmatrix} \nu & 0 & 0 \\ 0 & \nu & 0 \\ 0 & 0 & -2\nu \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix}$$

where n^0 is the monopole part and ν is a quadrupole anisotropy coefficient.

Putting $\dot{\mathbf{q}}$ in terms of $\mathbf{v}$, we can write out the components of momentum as

$$\mathbf{p} = (n^0 + 4\nu)\,\dot{\mathbf{q}} - 3\nu\dot{\mathbf{q}}^2\,\dot{\mathbf{q}} = (1 + \mathbf{v}^2)^{-3/2}[(n^0 + 4\nu) + (n^0 + \nu)\mathbf{v}^2]\mathbf{v}$$

For free propagation, the Hamilton equations and their solutions are:

$$\frac{d\mathbf{q}}{dz} = \mathbf{v} \Rightarrow \mathbf{q}(z) = \mathbf{q}(0) + z\mathbf{v}, \qquad \frac{d\mathbf{p}}{dz} = 0 \Rightarrow \mathbf{p}(z) = \mathbf{p}(0).$$

Although the solutions are apparently independent of the anisotropy of the medium (they are straight lines in space), the anisotropy is expressed through the relation between the ray momentum $\mathbf{p}$ and the ray direction $\mathbf{v}$. In isotropic media, the momentum vector is n times the direction vector and we can easily invert this particular case to [7]

$$\mathbf{v} = \frac{\mathbf{p}}{\sqrt{(n^0)^2 - \mathbf{p}^2}} = \frac{\mathbf{p}}{p_z}, \qquad |\mathbf{v}| = n^0 \tan\theta \qquad (\nu = 0).$$

In the more general uniaxial anisotropic case, to find a simple closed inversion, we expand this equation with a Taylor series in $(\mathbf{v}^2)^k\mathbf{v}$ for $k = 0, 1, 2, \ldots$ and propose a similar expansion of $\mathbf{v}$ in powers of $(\mathbf{p}^2)^k\mathbf{p}$. Equating the series we find the expansion coefficients

$$\begin{aligned}\mathbf{v}(\mathbf{p}) \;=\; & \frac{1}{n^0 + 4\nu}\,\mathbf{p} + \frac{\frac{1}{2}n^0 + 5\nu}{(n^0 + 4\nu)^4}\,\mathbf{p}^2\,\mathbf{p} + \frac{\frac{3}{8}(n^0)^2 + \frac{15}{2}n^0\nu + 51\nu^2}{(n^0 + 4\nu)^7}\,(\mathbf{p}^2)^2\,\mathbf{p} \\ & + \frac{\frac{5}{16}(n^0)^3 + \frac{75}{8}(n^0)^2\nu + 114n^0\nu^2 + 650\nu^3}{(n^0 + 4\nu)^{10}}\,(\mathbf{p}^2)^3\,\mathbf{p} + \cdots \;.\end{aligned}$$

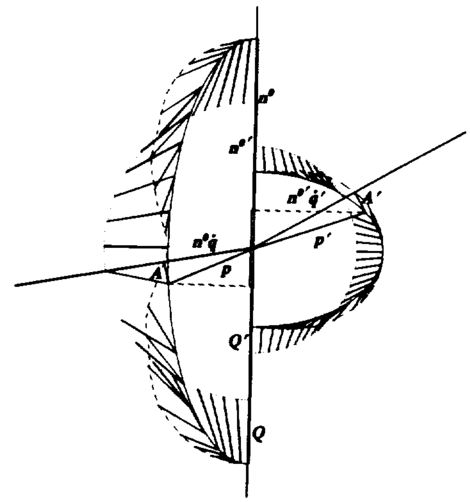


FIGURE 3. Descartes diagram to construct refraction angles between two anisotropic media.

The evolution Hamiltonian is then

$$\begin{aligned} H(\mathbf{p}) &= \mathbf{p}\cdot\mathbf{v} - (n^0 - 2\nu)\sqrt{1+\mathbf{v}^2} - 3\nu\,\frac{\mathbf{v}^2}{\sqrt{1+\mathbf{v}^2}} \\ &= -(n^0 - 2\nu) + \frac{1}{2n^e}\,\mathbf{p}^2 + \frac{n^0 + 10\nu}{8(n^e)^4}\,(\mathbf{p}^2)^2 + \frac{3(n^0)^2 + 60n^0\nu + 408\nu^2}{16(n^e)^7}\,(\mathbf{p}^2)^3 \\ &\quad + \frac{5(n^0)^3 + 150(n^0)^2\nu + 1824n^0\nu^2 + 10\,400\,\nu^3}{128(n^e)^{10}}\,(\mathbf{p}^2)^4 + \cdots\ , \end{aligned}$$

where $n^e = n^0 + 4\nu$ plays the role of an effective paraxial refractive index. In the isotropic case (when $\nu = 0$), this is the expansion of $-\sqrt{n^2 - \mathbf{p}^2}$, the well-known optical Hamiltonian for such a media [8].

6 Finite refraction

Let us consider the case when rays cross the flat interface $z = 0$ between two different aligned uniaxial quadrupole anisotropic media. Let their refractive indices in two half-spaces be $n(\dot{\vec{q}})$ and $n'(\dot{\vec{q}})$ with monopole parameters n^0 and $n^{0\prime}$, and quadrupole anisotropy coefficients ν and ν', respectively. The refraction law claims that the projection of the momentum vector on the refracting surface is *conserved*, which for our refracting surface gives $\mathbf{p} = \mathbf{p}'$. Generally, the incident and refracted rays are not coplanar with the surface normal. However, in the aligned uniaxial case both refractive indexes are axially symmetric (under rotations around z-axis) and the two anisotropic vectors are coplanar with the suface normal (z-axis). Refraction in our case is thus coplanar.

Using 'ruler, compass and plotter' on the plane Figure 3, we construct the Descartes diagram for the point at the interface joining two 'half Descartes diagrams' and matching the length of the momentum vectors $\mathbf{p}$ and $\mathbf{p}'$ on the interface. To find the angle of refraction θ' in terms of the angle of incidence θ, we construct $\dot{\mathbf{q}}'(\dot{\mathbf{q}}; n^0, \nu; n^{0\prime}, \nu')$ expanded in series of sines, and find

$$\sin\theta' = \frac{n^e}{n^{e'}}\sin\theta + \frac{3}{n^{e'}}\left[\left(\frac{n^e}{n^{e'}}\right)\nu' - \nu\right]\sin^3\theta + \frac{27}{(n^{e'})^2}\left(\frac{n^e}{n^{e'}}\right)^2\nu'\left[\left(\frac{n^e}{n^{e'}}\right)^3\nu' - \nu\right]\sin^5\theta$$

$$-\frac{81}{(n^{e'})^3}\left(\frac{n^e}{n^{e'}}\right)\nu'\left[\left(\frac{n^e}{n^{e'}}\right)^3\nu'-\nu\right]\left[\left(4\frac{n^e}{n^{e'}}\right)^3\nu'-\nu\right]\sin^7\theta+\cdots \ .$$

The first summand is the very well know law of sines (Ibn Sahl–Snell law [5]); it is here also the *paraxial approximation* with the ratio of effective refractive coefficients. The succeding terms are corrections of orders ν^k and $\sin^{2k+1}\theta$ due to anisotropy.

7 The roots of refraction

We consider now the ray transformation due to refraction at a smooth surface $S(\vec{q}) = \zeta(\mathbf{q}) - z = 0$ between two general anisotropic, homogeneous media $n(\vec{q})$ and $n'(\vec{q})$. The rays in the first and in the second media are given correspondingly by the equations

$$\begin{aligned} \mathbf{q}(z) = \mathbf{q} + z\mathbf{v}, \quad \mathbf{p}(z) = \mathbf{p}, \quad z < \bar{z}, \\ \mathbf{q}'(z) = \mathbf{q}' + z\mathbf{v}', \quad \mathbf{p}'(z) = \mathbf{p}', \quad z > \bar{z}, \end{aligned}$$

where we have indicated the point of impact at the refracting surface by bars $\bar{\vec{q}} = (\bar{\mathbf{q}},\ \bar{z} = \zeta(\bar{\mathbf{q}}))$. We can formally consider the second pair of equations also on the left of the refracting surface, $z < \bar{z}$. It allows to parametrize the rays behind the surface by the coordinate $\mathbf{q}'$ and momentum $\mathbf{p}'$ on the same screen $z = 0$; $\mathbf{v}$ and $\mathbf{v}'$ are the two ray directions on the screen. Thus, the point of impact coordinates can be written in two ways:

$$\mathbf{q}(\bar{z}) = \mathbf{q} + \zeta(\bar{\mathbf{q}})\mathbf{v} = \bar{\mathbf{q}} = \mathbf{q}' + \zeta(\bar{\mathbf{q}})\mathbf{v}' = \mathbf{q}'(\bar{z}).$$

This is the *first root equation* of refraction [9]; it is an implicit equation for $\bar{\mathbf{q}}$.

The *second root equation* follows from the conservation of the tangential component of momentum and implies the refraction law. If the normal to the surface S is denoted by $\vec{\nabla}S(\vec{q}) = (\zeta_x, \zeta_y, -1) \equiv (\mathbf{\Sigma}(\mathbf{q}), -1)$ then we have $(\vec{p} - \vec{p}') \times \vec{\nabla}S(\vec{q}) = 0$. As we know, the momentum vector has components $\vec{p} = (p_x, p_y, p_z) = (\mathbf{p}, -H)$. Denoting the Hamiltonians before and after the refracting suface as H and H' we can rewrite the last equation containing the vector product as

$$\mathbf{p} - H(\mathbf{p})\mathbf{\Sigma}(\bar{q}) = \bar{\mathbf{p}} = \mathbf{p}' - H'(\mathbf{p}')\mathbf{\Sigma}(\bar{q}).$$

This is the second root equation determaning explicitly $\bar{p}$ once $\bar{q}$ has been found.

We have thus determined the root transformation for generic surfaces $S = \zeta(\mathbf{q}) - z = 0$ between homogeneous, anisotropic media. On optical phase space the root transformation is

$$\begin{aligned} \mathcal{R}_{n;\zeta} : \mathbf{q} &\mapsto \bar{\mathbf{q}} = \mathbf{q} + \mathbf{v}(\mathbf{p})\,\zeta(\bar{\mathbf{q}}), \\ \mathcal{R}_{n;\zeta} : \mathbf{p} &\mapsto \bar{\mathbf{p}} = \mathbf{p} - H(\mathbf{p})\,\mathbf{\Sigma}(\bar{\mathbf{q}}), \end{aligned}$$

where $\mathbf{v}(\mathbf{p})$ and $H(\mathbf{p})$ contain the refractive index function $n(\vec{q})$. From our construction follows that the refracting surface transformation

$$\mathcal{S}_{n,n';\zeta} : (\mathbf{q}, \mathbf{p}) \mapsto (\mathbf{q}', \mathbf{p}')$$

thus factorizes into the product of the root transformation in the first medium and the inverse root transformation in the second medium, $\mathcal{S}_{n,n';\zeta} = \mathcal{R}_{n;\zeta}\,(\mathcal{R}_{n';\zeta})^{-1}$. When the surface S is a

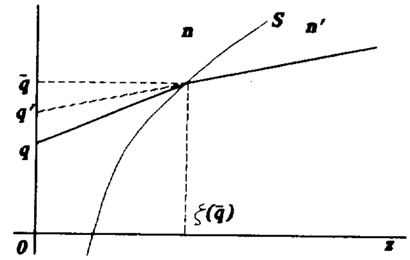


FIGURE 4. Refraction at a surface is a map between phase space points $(\mathbf{q},\mathbf{p})$ and $(\mathbf{q}',\mathbf{p}')$. This transformation visibly factors into transformations back and forth from the point of impact $\mathbf{q}$ on the surface $z=\zeta(\mathbf{q})$.

$z=$ constant plane, the second root transformation is simple free flight by generic z. The root transformation is illustrated in figure 4.

Let us consider explicitly the example of the symmetrical surface under rotations around z-axis

$$\zeta(\mathbf{q}) = \zeta_2\mathbf{q}^2 + \zeta_4\mathbf{q}^4 + \cdots.$$

The refraction by such a surface is determined to third aberration order as [9]

$$\begin{aligned}
\mathbf{q}' &= \mathbf{q} - \zeta_2\left(\frac{1}{n^{0\prime}+4\nu'} - \frac{1}{n^0+4\nu}\right)\mathbf{q}^2\mathbf{p} + 2\zeta_2^2\,\frac{[n^{0\prime}-2\nu']-[n^0-2\nu]}{n^{0\prime}+4\nu}\,\mathbf{q}^2\mathbf{q},\\
\mathbf{p}' &= \mathbf{p} + 2\zeta_2([n^{0\prime}-2\nu']-[n^0-2\nu])\mathbf{q} + \zeta_2\left(\frac{1}{n^{0\prime}+4\nu'} - \frac{1}{n^0+4\nu}\right)\mathbf{p}^2\mathbf{q}\\
&\quad -4\zeta_2^2\,\frac{[n^{0\prime}-2\nu']-[n^0-2\nu]}{n^{0\prime}+4\nu}\,\mathbf{p}\cdot\mathbf{q}\mathbf{q} - 2\zeta_2^2\,\frac{[n^{0\prime}-2\nu']-[n^0-2\nu]}{n^{0\prime}+4\nu}\,\mathbf{q}^2\mathbf{p}\\
&\quad +4\left(\zeta_2^3\,\frac{([n^{0\prime}-2\nu']-[n^0-2\nu])^2}{n^{0\prime}+4\nu} - \zeta_4\left([n^{0\prime}-2\nu']-[n^0-2\nu]\right)\right)\mathbf{q}^2\mathbf{q}.
\end{aligned}$$

The paraxial part of the transformation is recognizably that of a quadratic surface.

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13. ABSTRACT *(Maximum 200 words)*

The second International Workshop on Harmonic Oscillators was held at the Hotel Hacienda Cocoyoc March 23-25, 1994. The Workshop was attended by 67 participants; there were 10 invited lectures, 30 plenary oral presentations, 15 posters, and plenty of discussion divided into the five sessions of this volume. The organizing committee was asked by the chairmen of several Mexican funding agencies exactly what was meant by harmonic oscillators, and for what purpose the new research could be useful. Harmonic oscillator, in group theory, is a name for a family of mathematical models based on the theory of Lie algebras and groups, with applications in a growing range of physical theories and technologies: molecular, atomic, nuclear, and particle physics; quantum optics; and communication theory. The Harmonic Oscillators II Workshop was funded and organized through the Centro Internacional de Fisica y Matematicas Aplicadas (CIFMA). The Cuernavaca Center adds to the existing networks initiated by the Centro Latino Americano de Fisica, in Brazil, and the Centro Internacional de Fisica, in Colombia. The First Harmonic Oscillators meeting was held at the College Park campus of the University of Maryland (March 25-28, 1992).

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